Large Amplitude Free Vibrations of Shells of Variable Thickness—A New Approach

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Large-amplitude free vibrations of thin elastic shallow spherical and cylindrical shells of nonuniform thickness have been investigated following a new approach. Numerical results for movable as well as for immovable edge conditions are given in tabular forms and compared with other results.

Nomenclature

A	= nondimensional amplitude
D	= modulus of rigidity
E	= Young's modulus
e,e_2	= first and second strain invariants of the middle surface
$\bar{e}_1,\bar{e}_2$	= newly defined strain invariants
h	= shell thickness
k	= curvature at the middle surface, in case of cylindrical shell
R	= radius of the base of the spherical shell
R_0	= radius of the spherical shell
r,θ	= polar coordinates
T	= kinetic energy
V,V_1	= potential energy of spherical and cylindrical shells, respectively
$w_0(t)$	= unspecified function of time
x,y	= Cartesian coordinates
α	= normalized constant of integration
$\epsilon_x,\epsilon_y,\gamma_{xy}$	= strain components in Cartesian coordinates in middle surface
ρ	= density of the shell material
ν	= Poisson's ratio of the shell material

Introduction

PLATES and shells of nonuniform thickness are sometimes encountered in the design of machine parts, and their stress analyses are imperative to design engineers. As far as it is known by these authors, no paper could be located in which nonlinear dynamic behaviors of shells of variable thickness have been studied.

It is well known that Berger[1] presented a simplified approach to analyze the large deflection of plates. Berger's method is based on the neglect of e_2, the second invariant of the middle surface strain. This well-known method has been utilized successfully by many authors in their problems. Nowinski and Ohnabe[2] pointed out that although Berger's method is advantageous for its simplicity, it fails miserably for movable edge conditions. Nash and Modeer[3] extended Berger's method to analyze the nonlinear behavior of a thin, elastic, shallow spherical shell and achieved satisfactory results. Nash and Modeer's results have certain limitations, namely, the analyses have been carried out for immovable edge conditions only. Nowinski and Ismail[4] followed Berger's line of thought and analyzed quite elegantly the large deflections of cylindrical shell panels. Bhattacharya[5] followed Berger's method and investigated the variations of nondimensional frequency ratios with the dimensionless amplitudes of thin, shallow, cylindrical shells. Ramachandran,[6] using Marguerre's shallow shell equations, analyzed successfully the large-amplitude vibrations of shallow spherical shells. This method of solution is laborious, involving as it does the solution of the coupled form of the partial differential equations. All these investigations are confined to shells of uniform thickness only. Recently Sinharay and Banerjee[7] have offered a modified strain energy expression to study the nonlinear dynamic behavior of spherical and cylindrical shells and also to overcome the difficulties arising from Berger's approach in case of movable edges. They derived the differential equations in a decoupled form for each shell. The authors' approach, in which the differential equations are decoupled, has the advantage of greater simplicity over Berger's or more general classical approaches. Moreover, unlike Berger's approximation, this new approach offers sufficiently accurate results for both movable and immovable edge conditions from a single differential equation. As far as the authors know, no paper has yet been devoted to analyzing the nonlinear dynamic behavior of different shells of variable thickness. Thus, the results of the present study are new. Comparison between two sets of results has been made, applying the technique offered by Berger and by the present study of the same problem. It has been observed that the results of the present study for shells of constant thickness[7] are in excellent agreement with those obtained by Bhattacharya and by Ramachandran.

The present study offers a new set of differential equations for shells of variable thickness where the authors use their modified strain energy expressions proposed for shells of constant thickness.[7] Keeping in mind that the shell thickness is a variable quantity, a new set of partial differential equations for spherical as well as for cylindrical shells has been derived using Euler's variational equations and Hamilton's principle to the sum of these modified energy expressions and the kinetic energy of the shell. The differential equations for the unknown time function thus obtained for each shell have been solved by the well-known Galerkin's error-minimizing technique. In particular, the nonlinear dynamic behaviors of 1) clamped thin elastic spherical shells with particular form of thickness variation, and 2) simply supported cylindrical shells with particular form of thickness variation have been investigated both for movable as well as for immovable edge conditions. Numerical results are given in tabular form for each shell for different values of thickness variation parameters and also for various geometries. The results are compared with other results. A discussion on the observations of these results has also been carried out.

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Formation and Solution of Differential Equations for a Thin Spherical Shell

Let us consider the free vibration of a thin elastic shallow spherical shell of nonuniform thickness with clamped edges. The coordinate system to be employed is as given in Fig. 1. The normal component of the displacement of the middle surface of the shell is denoted by w, considered to be positive from the concave to convex direction. The radial displacement of a point in the middle surface is denoted by u, measured meridionally away from the axis of symmetry. The elevation of the middle surface of the shell above the base plane is denoted by $z = z(r)$.

As referred to geometry,[3] we have

$$Z = \frac{R^2}{2R_0}\left(1 - \frac{r^2}{R^2}\right) \tag{1}$$

from which dz/dr is calculated, R is the radius of the base, and R_0 is the radius of the shell.

The total potential energy due to bending and stretching may be written as[3]

$$V = \int\int \frac{D}{2}\left[(\nabla^2 w)^2 - \frac{2(1-\nu)}{r}\frac{dw}{dr}\frac{d^2w}{dr^2} + \frac{12}{h^2}\{e^2 - 2(1-\nu)e_2\}\right] r dr d\theta$$

where

$$e = \frac{du}{dr} + \frac{u}{r} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \frac{dw}{dr}\frac{dz}{dr}$$

$$e_2 = \frac{u}{r}\frac{du}{dr} + \frac{u}{2r}\left(\frac{dw}{dr}\right)^2 + \frac{u}{r}\frac{dw}{dr}\frac{dz}{dr}$$

with meanings as given in the Nomenclature.

Let us now rewrite the potential energy in the following form:

$$V = \int\int \frac{D}{2}\left[(\nabla^2 w)^2 - \frac{2(1-\nu)}{r}\frac{dw}{dr}\frac{d^2w}{dr^2}\right] r dr d\theta + \int\int \frac{6D}{h^2}\left[\bar{e}_1^2 + \lambda\left\{\frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \frac{dw}{dr}\frac{dz}{dr}\right\}^2\right] r dr d\theta \tag{2}$$

Here, the term $(1-\nu^2)u^2/r^2$ has been replaced by

$$\lambda\left\{\frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \frac{dw}{dr}\frac{dz}{dr}\right\}^2$$

and

$$\bar{e}_1 = \frac{du}{dr} + \frac{\nu u}{r} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \frac{dw}{dr}\frac{dz}{dr} \tag{3}$$

and λ is a factor depending on the Poisson's ratio of the shell material.[7] In the case of a shell, the radial stretching of the shell is proportional to $½[(dw/dr)^2 + (dw/dr) \times (dz/dr)]$. This is certainly reasonable because the contribution of the term $½[(dw/dr)^2 + (dw/dr)(dz/dr)]$ in the expression for ϵ_r, where $\epsilon_r = (du/dr) + ½[(dw/dr)^2 + (dw/dr)(dz/dr)]$ is greater than that of du/dr in bending and under any type of loading and under any boundary conditions. The extra strain imposed by bending for a shell is represented by the term $½[(dw/dr)^2] + (dw/dr)(dz/dr)$. Therefore, the tangential strain $\epsilon_t = u/r$ can be assumed to be proportional to $½[(dw/dr)^2] + (dw/dr)(dz/dr)$. The constant of proportionality is a function of ν, because ϵ_r and ϵ_θ are mutually perpendicular. The kinetic energy of the shell is given by $T = \rho/2\int\int h(\dot{u}^2 + \dot{w}^2) r dr d\theta$.

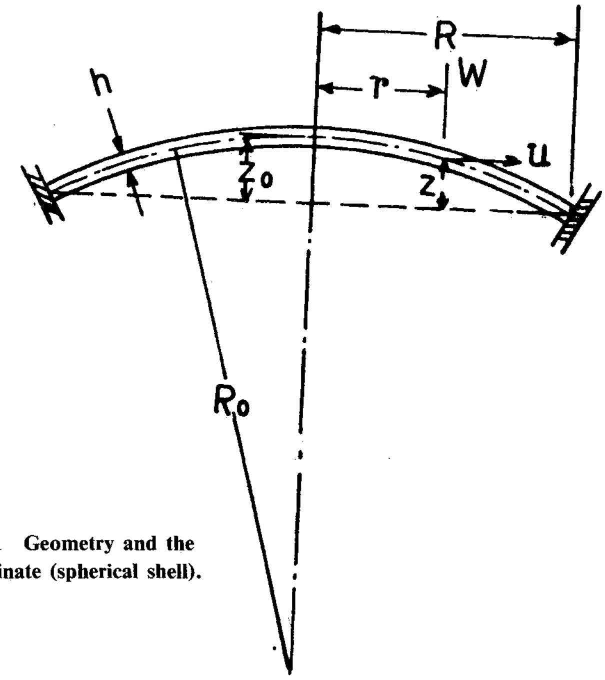

Fig. 1 Geometry and the coordinate (spherical shell).

Under the above assumptions, decoupling of the differential equation is possible though the shell thickness is a variable quantity.

Forming the Lagrangian function $L = T - V$, applying Hamilton's principle, and using Euler's variational equations, the following set of differential equations is obtained, neglecting the in-plane inertia, in a decoupled form,

$$hr^{1-\nu}\bar{e}_1 = \frac{\alpha^2}{12}f(t) \tag{4}$$

where α is a constant and $f(t)$ is a function of time

$$h^3\nabla^4 w - \alpha^2 f(t) r^{\nu-1}\nabla^2 w - \alpha^2 f(t) r^{\nu-2}\left[(\nu-1)\frac{dw}{dr} + \nu\frac{dz}{dr} + r\frac{d^2z}{dr^2}\right] + 3h^2\left[\frac{dh}{dr}\left\{2\frac{d^3w}{dr^3} + \frac{\nu+2}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr}\right\} + \left\{\frac{2}{h}\left(\frac{dh}{dr}\right)^2 + \frac{d^2h}{dr^2}\right\}\left\{\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right\}\right]$$
$$- 12\lambda h\left[\left(\frac{d^2w}{dr^2} + \frac{d^2z}{dr^2}\right)\left\{\frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \frac{dw}{dr}\frac{dz}{dr}\right\} + \left(\frac{dw}{dr} + \frac{dz}{dr}\right)\left\{\frac{d^2w}{dr^2}\left(\frac{dw}{dr} + \frac{dz}{dr}\right) + \frac{dw}{dr}\frac{d^2z}{dr^2}\right\} + \left(\frac{dw}{dr} + \frac{dz}{dr}\right)\left\{\frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \frac{dw}{dr}\frac{dz}{dr}\right\} \times \left(\frac{1}{r} + \frac{1}{h}\frac{dh}{dr}\right)\right] + \frac{\rho h}{L_1}\frac{d^2w}{dt^2} = 0 \tag{5}$$

where $L_1 = E/12(1-\nu^2)$.

For a movable edge $\alpha = 0$, as the in-plane radial stress vanishes at the boundary.

Let us assume

$$w = Aw_0(t)\left(1 - \frac{r^2}{R^2}\right)^2 \tag{6}$$

where A stands for nondimensional amplitude and $w_0(t)$ is the unspecified function of time. Equation (6) clearly satisfies the clamped edge boundary condition.

As an illustration, we assume

$$h = h_0\left(1 + \frac{kr}{R}\right) \tag{7}$$

(k being a parameter) to be the thickness variation, where h_0 is the central thickness.

To solve Eq. (4), let us put Eqs. (6) and (7) in Eq. (4), and integrate over the area of the shell. We get

$$\frac{\alpha^2 f(t)}{12} = h_0\left[\frac{8Aw_0}{R_0}\frac{1}{(\nu+3)(\nu+5)}R^{1-\nu} + \frac{64A^2w_0^2}{(\nu+3)(\nu+5)(\nu+7)}\bar{R}^{\nu-1}\right] \Big/ \sum_{s=0}^{\infty}\frac{(-k)^s}{2\nu+s} \tag{8}$$

To solve Eq. (5), let us use Galerkin's error minimizing technique. Putting Eqs. (6) and (7) in Eq. (5), using Eq. (8), and applying Galerkin's technique, we get the following equation, determining the time function $w_0(\tau)$ in the following form:

$$\frac{d^2(w_0/h_0)}{d\tau^2} + \mu_1\frac{w_0}{h_0} + \mu_2\left(\frac{w_0}{h_0}\right)^2 A + \mu_3\left(\frac{w_0}{h_0}\right)^3 A^2 = 0 \tag{9}$$

where

$$\mu_1 = \left[\left\{\frac{32}{3} + k\left(\frac{812-128\nu}{35}\right) + k^2(20-4\nu) + \frac{k^3(1856-384\nu)}{315}\right\} + \xi^2\left\{\frac{768}{(\nu+3)^2(\nu+5)^2}\left(1\Big/\sum_{s=0}^{\infty}\frac{(-k)^s}{2\nu+s}\right) + \lambda\left(\frac{16}{5} + k\frac{512}{231}\right)\right\}\right]\Big/\left(\frac{1}{10} + \frac{128k}{3465}\right)$$

$$\mu_2 = \left[\frac{18432}{(\nu+3)^2(\nu+5)^2(\nu+7)}\left(1\Big/\sum_{s=0}^{\infty}\frac{(-k)^s}{2\nu+s}\right) + 1152\lambda\left(\frac{1}{120} + \frac{80k}{15015}\right)\right]\xi\Big/\left(\frac{1}{10} + \frac{128k}{3465}\right)$$

$$\mu_3 = \left[\frac{98304}{(\nu+3)^2(\nu+5)^2(\nu+7)^2}\left(1\Big/\sum_{s=0}^{\infty}\frac{(-k)^s}{2\nu+s}\right) + 256\lambda\left(\frac{1}{35} + \frac{256k}{15015}\right)\right]\Big/\left(\frac{1}{10} + \frac{128k}{3465}\right)$$

$$\xi = \frac{R^2}{R_0h_0}, \qquad \tau = t\left[\frac{L_1h_0^2}{\rho R^4}\right]^{1/2} \tag{10}$$

For movable edges,

$$\mu_1 = \left[\left\{\frac{32}{3} + k\left(\frac{812-128\nu}{35}\right) + k^2(20-4\nu) + k^3\frac{(1856-384\nu)}{315}\right\} + \lambda\xi^2\left(\frac{16}{5} + \frac{512k}{231}\right)\right]\Big/\left(\frac{1}{10} + \frac{128k}{3465}\right)$$

$$\mu_2 = 1152\lambda\left(\frac{1}{120} + \frac{16k}{3003}\right)\xi\Big/\left(\frac{1}{10} + \frac{128k}{3465}\right)$$

$$\mu_3 = 256\lambda\left(\frac{1}{35} + \frac{256k}{15015}\right)\Big/\left(\frac{1}{10} + \frac{128k}{3465}\right) \tag{11}$$

The values of λ have been obtained from the condition of minimum potential energy, and the values are $\lambda = 2\nu^2$ for clamped edges and $\lambda = \nu^2$ for simply supported edges.[7]

If the initial conditons are $w_0/h_0 = 1$, $[d(w_0/h_0)]/d\tau = 0$ at $\tau = 0$, then the solutions can be written as

$$\frac{\omega^*}{\omega} = \left[1 + A^2\left\{\frac{3}{4}\frac{\mu_3}{\mu_1} - \frac{5}{6}\left(\frac{\mu_2}{\mu_1}\right)^2\right\}\right]^{1/2}$$

where ω^* and ω are the nonlinear and linear frequencies, respectively.

Numerical Results

Table 1 presents a comparative study of the variation of nonlinear frequency ω^* with the dimensionless amplitudes A for spherical caps with immovable edges for various geometries ($\xi = 0.5$, 1.0) with different thickness variation parameters ($k = 0$, 0.1, 0.2, 0.3) using $\nu = 0.3$, $\lambda = 2\nu^2$. In Table 1 results of the present study are compared with those obtained by Berger's approach.

Table 2 presents the variation of nonlinear frequency vs dimensionless amplitudes for spherical caps with movable edges for various geometries with different thickness variation parameters, using $\nu = 0.3$, $\lambda = 2\nu^2$. Berger's approach as summarized in Table 2 gives unsound results for movable edges.

Formation of Differential Equation for Cylindrical Shell and Solution

Let us consider a class of thin shallow translational shells bounded by $x = \pm a$, $y = \pm b$, the middle surface of which is defined in Cartesian coordinates by

$$z(x,y) = \tfrac{1}{2}(k_3x^2 + k_4y^2) \tag{12}$$

k_3/k_4 being the curvature parameter of the surface. For definiteness, we assume that all the edges are simply supported.

Table 1 Comparative study of spherical shells with immovable edges

		ω^*							
		$k=0$		$k=0.1$		$k=0.2$		$k=0.3$	
ξ	A	Present	Berger	Present	Berger	Present	Berger	Present	Berger
1.0	0.25	11.312	11.547	12.155	12.287	13.036	13.060	13.950	13.867
	0.5	11.378	11.547	12.245	12.315	13.143	13.113	14.070	13.939
	0.75	11.485	11.547	12.393	12.362	13.318	13.201	14.264	14.059
	1.0	11.633	11.547	12.595	12.428	13.560	13.322	14.533	14.227
0.5	0.25	10.662	10.729	11.539	11.490	12.454	12.290	13.401	13.122
	0.5	10.911	10.973	11.781	11.735	12.687	12.534	13.625	13.353
	0.75	11.310	11.368	12.173	12.132	13.065	12.928	13.987	13.754
	1.0	11.856	11.898	12.866	12.666	13.576	13.463	14.507	14.285

If V_1 represents the potential energy of the shell due to bending and stretching, and T denotes the kinetic energy, then, neglecting the effects of transverse shear and rotatory inertia, we have

$$V_1 - T = \iint \frac{D}{2}\left[(\nabla^2 w)^2 - 2(1-\nu)\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\} + \frac{12}{h^2}\{e^2 - 2(1-\nu)e_2\}\right] r\,\mathrm{d}r\mathrm{d}\theta - \frac{\rho}{2}\iint h(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\mathrm{d}x\mathrm{d}y \tag{13}$$

in which

$$D = [Eh^3/12(1-\nu^2)], \quad e = \epsilon_x + \epsilon_y, \quad e_2 = \epsilon_x\epsilon_y - \tfrac{1}{4}\gamma_{xy}^2 \tag{14}$$

and

$$\epsilon_x = \frac{\partial u}{\partial x} - k_1 w + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$\epsilon_y = \frac{\partial v}{\partial y} - k_2 w + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \tag{15}$$

$$2k = k_1 + k_2$$

Equation (15) is due to von Kármán and Tsien.[8]

In the foregoing, t represents time and u,v, are the tangential displacement components, and w is the displacement component normal to the shell, in a manner defined by Flügge and Conrad.[9]

Table 2 Present study of spherical shells with movable edges

		ω^*			
ξ	A	$k=0$	$k=0.1$	$k=0.2$	$k=0.3$
1.0	0.25	10.625	11.510	12.431	13.381
	0.5	10.665	11.576	12.496	13.444
	0.75	10.805	11.684	12.601	13.547
	1.0	10.958	11.835	12.749	13.688
0.5	0.25	10.425	11.317	12.248	13.208
	0.5	10.508	11.391	12.324	13.279
	0.75	10.645	11.512	12.448	13.398
	1.0	10.835	11.679	12.620	13.558

Let us now rewrite $V_1 - T$ in the following form:

$$V_1 - T = \iint \frac{D}{2}\left[(\nabla^2 w)^2 - 2(1-\nu)\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\}\right]\mathrm{d}x\mathrm{d}y - \frac{\rho}{2}\iint h(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\mathrm{d}x\mathrm{d}y + 6\iint \frac{D}{h^2}\left[\bar{e}_2^2 + \lambda\left\{\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - (k_1 + k_2)w\right\}^2\right]\mathrm{d}x\mathrm{d}y \tag{16}$$

where

$$\bar{e}_2 = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - k_1 w + \nu\frac{\partial v}{\partial y} + \frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2 - k_2 w \tag{17}$$

By the same reasoning as given in the case of a spherical shell, $e^2 - 2(1-\nu)e_2$ is replaced in Cartesian coordinates by $\bar{e}_2^2 + \lambda\{\tfrac{1}{2}(\partial w/\partial x)^2 + \tfrac{1}{2}(\partial w/\partial y)^2 - (k_1 + k_2)w\}^2$, in this case of a cylindrical shell. Applying Hamilton's principle on the Lagrangian function and using Euler's variational equations, remembering that h is a variable quantity, the following set of decoupled differential equations is obtained:

$$h\bar{e}_2 = \frac{A_1}{12}f(t) \tag{18}$$

where A_1 is constant and $f(t)$ is a function of time; and

$$h^3\nabla^4 w + 6\nabla^2\left(\frac{\partial w}{\partial x}\right)h^2\frac{\mathrm{d}h}{\mathrm{d}x} + 3\left[2h\left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^2 + h^2\frac{\mathrm{d}^2 h}{\mathrm{d}x^2}\right]\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right] - A_1\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} + 2k\right) - 6\lambda h\left[\left\{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\}\left(\nabla^2 w + \frac{\partial w}{\partial x}\frac{1}{h}\frac{\partial h}{\mathrm{d}x} - 2k\right) - 4kw\left(\nabla^2 w + \frac{\partial w}{\partial x}\frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}x} + 2k\right) + 2\left\{\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} + \left(\frac{\partial w}{\partial y}\right)^2\frac{\partial^2 w}{\partial y^2}\right\}\right] + \frac{\rho h}{L}\frac{\partial^2 w}{\partial t^2} = 0 \tag{19}$$

Table 3 Comparitive study of square panels with immovable edges

		ω^*							
		$m=0$		$m=0.1$		$m=0.2$		$m=0.3$	
ρ	A	Present	Berger	Present	Berger	Present	Berger	Present	Berger
0.25	0.25	8.562	8.603	8.570	8.607	8.592	8.620	8.630	8.641
	0.5	8.716	8.886	8.723	8.890	8.745	8.900	8.780	8.916
	0.75	8.966	9.340	8.974	9.342	8.993	9.348	9.025	9.355
	1.0	9.305	9.940	9.313	9.940	9.329	9.940	9.356	9.937
1.25	0.25	35.026	35.015	34.971	34.960	34.808	34.795	34.532	34.514
	0.5	35.064	35.085	34.975	35.030	34.847	34.865	34.570	34.583
	0.75	35.172	35.201	35.073	35.146	34.909	34.980	34.632	34.697
	1.0	35.110	35.632	35.160	35.310	34.996	35.143	34.179	34.859
2.5	0.25	69.513	69.472	69.398	69.356	69.050	69.007	68.461	68.415
	0.5	69.527	69.507	69.412	69.391	69.064	69.041	68.475	68.449
	0.75	69.562	69.562	69.447	69.446	69.098	69.096	68.509	68.504
	1.0	69.603	69.646	69.488	69.529	69.140	69.179	68.550	68.586

where $L=E/12(1-\nu^2)$, $2k=k_1+k_2$. For movable edge $A_1=0$.

Here also we assume

$$h=h_0[1+(mx/a)], \qquad m<1 \tag{20}$$

(m is parameter) to be the thickness variations.

To solve Eq. (18), we integrate it, with the help of Eq. (20), over the area, i.e., from $x=-a$ to $x=+a$, from $y=-b$ to $y=+b$. Thus we get

$$A_1 f(t)=6mh_0\left[\frac{A^2 w_0^2 \pi^2}{8}\left(\frac{1}{a^2}+\frac{\nu}{b^2}\right)-\frac{32kAw_0}{\pi^2}\right]\Big/\log\frac{1+m}{1-m}$$

where we assume

$$w=Aw_0(t)\cos\frac{\pi x}{2a}\cos\frac{\pi y}{2b} \tag{22}$$

as first term approximation, A is nondimensional amplitude and $w_0(t)$ is the unspecified function of time. We note that Eq. (22) clearly satisfies the boundary condition of simple support.

It is to be noted that we are interested only in the normal displacement w and, therefore, the in-plane displacements u and v can easily be eliminated by considering suitable expressions for these displacements compatible with boundary conditions and subsequent integration of Eq. (17) over the required limits.

Substituting Eqs. (22) and (20) into Eq. (19), using Eq. (21), and applying Galerkin's error-minimizing technique, we get the following equation, determining the time function $w_0(\tau)$ in the following form:

$$\frac{d^2(w_0/h_0)}{d\tau^2}+\mu_1\left(\frac{w_0}{h_0}\right)+\mu_2\left(\frac{w_0}{h_0}\right)^2 A+\mu_3\left(\frac{w_0}{h_0}\right)^3 A^2=0 \tag{23}$$

where

$$\mu_1=\frac{3m^2\pi^2}{4}\left\{1+(2-\nu)\frac{a^2}{b^2}\right\}+\frac{\pi^4}{16}\left(1+\frac{a^2}{b^2}\right)^2 \times\left\{1+m^2\left(1-\frac{6}{\pi^2}\right)\right\}+\left(\frac{a^2}{Rh_0}\right)^2 \times\left[12\lambda+\frac{1536m}{\pi^4\log[(1+m)/(1-m)]}\right]$$

$$-\mu_2=\left(\frac{a^2}{Rh_0}\right)\left[\frac{m}{\log[1+m)/(1-m)]}36\left(1+\nu\frac{a^2}{b^2}\right)+16\lambda\left(1+\frac{a^2}{b^2}\right)\right]$$

$$\mu_3=\frac{3\pi^4}{16}\left(1+\nu\frac{a^2}{b^2}\right)^2\frac{m}{\log[(1+m)/(1-m)]}+\frac{3\lambda\pi^4}{128}\left[9+9\frac{a^4}{b^4}+2\frac{a^2}{b^2}+\frac{24m}{\pi^2}\right] \tag{24}$$

and

$$\tau=t\left[\frac{Lh_0^2}{\rho a^4}\right]^{1/2}$$

$\rho=a^2/Rh_0$; as in the case of circular cylindrical shell, $1/(2k)$ reduces to R, the radius of the shell.

Here also, when $w_0/h_0=1$, $[d(w_0/h_0)/(d\tau)]=0$ at $\tau=0$, the solutions are

$$\frac{\omega^*}{\omega}=\left[1+A^2\left\{\frac{3}{4}\frac{\mu_3}{\mu_1}-\frac{5}{6}\left(\frac{\mu_2}{\mu_1}\right)^2\right\}\right]^{1/2} \tag{25}$$

When the edges are movable,

$$\mu_1=\frac{3m^2\pi^2}{4}\left\{1+(2-\nu)\frac{a^2}{b^2}\right\}+\frac{\pi^4}{16}\left(1+\frac{a^2}{b^2}\right)^2 \times\left\{1+m^2\left(1-\frac{6}{\pi^2}\right)\right\}+12\lambda\left(\frac{a^2}{Rh_0}\right)^2$$

$$\mu_2=-16\lambda\left(\frac{a^2}{Rh_0}\right)\left(1+\frac{a^2}{b^2}\right)$$

$$\mu_3=\frac{3\lambda\pi^4}{128}\left[9+9\frac{a^4}{b^4}+2\frac{a^2}{b^2}+\frac{24m}{\pi^2}\right] \tag{26}$$

Numerical Results: Square Panels

In this case, Eq. (24) reduces to

$$\mu_1=\frac{3m^2\pi^2}{4}(3-\nu)+\frac{\pi^4}{4}\left[1+m^2\left(1-\frac{6}{\pi^2}\right)\right]+\left[\frac{1536m}{\log[(1+m)/(1-m)]}+12\lambda\right]\left(\frac{a^2}{Rh_0}\right)^2$$

$$\mu_2=-\left(\frac{a^2}{Rh_0}\right)\left[36(1+\nu)\frac{m}{\log[(1+m)/(1-m)]}+32\lambda\right]$$

$$\mu_3=\frac{3\pi^4(1+\nu)^2}{16}\frac{m}{\log[(1+m)/(1-m)]}+\frac{3\lambda\pi^4}{128}\left(20+\frac{24m}{\pi^2}\right) \tag{27}$$

Table 3 presents a comparative study of the variation of nonlinear frequency ω^* vs dimensionless amplitudes A for simply supported square panels with immovable edges for various geometries ($\rho=0.25$, 1.25, 2.5) with different thickness variation parameters ($m=0$, 0.1, 0.2, 0.3) using $\nu=0.3$, $\lambda=\nu^2$. In this table, results of the present study are compared with those obtained by Berger's approximation.

In case of movable edges, Eq. (26) reduces to

$$\mu_1=\frac{3m^2\pi^2}{4}(3-\nu)+\frac{\pi^4}{4}\left\{1+m^2\left(1-\frac{6}{\pi^2}\right)\right\}+12\lambda\left(\frac{a^2}{Rh_0}\right)^2$$

$$\mu_2=-32\lambda\left(\frac{a^2}{Rh_0}\right)$$

$$\mu_3=\frac{3\lambda\pi^4}{32}\left(5+\frac{6m}{\pi^2}\right) \tag{28}$$

Table 4 Present study of square panels with movable edges

		ω^*			
ρ	A	$m=0$	$m=0.1$	$m=0.2$	$m=0.3$
0.25	0.25	4.961	4.990	5.079	5.223
	0.5	5.019	5.049	5.137	5.280
	0.75	5.114	5.143	5.232	5.373
	1.0	5.244	5.274	5.361	5.501
1.25	0.25	5.119	5.148	5.233	5.373
	0.5	5.168	5.197	5.283	5.422
	0.75	5.248	5.278	5.364	5.503
	1.0	5.358	5.390	5.475	5.614
2.5	0.25	5.586	5.661	5.692	5.820
	0.5	5.615	5.690	5.722	5.851
	0.75	5.662	5.739	5.771	5.902
	1.0	5.727	5.806	5.840	5.973

Table 5 Comparative study of infinitely long panels with immovable edges

		ω^*							
		$m=0$		$m=0.1$		$m=0.2$		$m=0.3$	
ρ	A	Present	Berger	Present	Berger	Present	Berger	Present	Berger
0.25	0.25	7.392	7.382	7.388	7.378	7.374	7.364	7.346	7.341
	0.5	7.491	7.464	7.488	7.460	7.477	7.446	7.444	7.422
	0.75	7.654	7.600	7.651	7.595	7.636	7.580	7.604	7.554
	1.0	7.876	7.786	7.873	7.781	7.859	7.764	7.825	7.735
1.25	0.25	34.760	34.736	34.703	34.678	34.530	34.506	34.236	34.212
	0.5	34.781	34.753	34.724	34.695	34.551	34.523	34.256	34.229
	0.75	34.816	34.781	34.758	34.723	34.585	34.551	34.291	34.718
	1.0	34.864	34.823	34.807	34.765	34.637	34.592	34.342	34.297
2.5	0.25	69.382	69.326	69.265	69.209	68.912	68.856	68.314	68.258
	0.5	69.389	69.340	69.272	69.223	68.919	68.870	68.321	68.272
	0.75	69.403	69.354	69.286	69.273	68.933	68.884	68.341	68.285
	1.0	69.431	69.368	69.313	69.251	68.960	68.897	68.362	68.299

Table 6 Present study of infinitely long panels with movable edges

		ω^*			
ρ	A	$m=0$	$m=0.1$	$m=0.2$	$m=0.3$
0.25	0.25	2.498	2.519	2.577	2.655
	0.5	2.549	2.571	2.629	2.707
	0.75	2.632	2.655	2.713	2.792
	1.0	2.743	2.768	2.827	2.906
1.25	0.25	2.800	2.818	2.870	2.942
	0.5	2.833	2.854	2.907	2.979
	0.75	2.891	2.912	2.967	3.041
	1.0	2.969	2.993	3.049	3.125
2.5	0.25	3.588	3.602	3.642	3.699
	0.5	3.602	3.618	3.659	3.717
	0.75	3.626	3.643	3.687	3.747
	1.0	3.658	3.678	3.725	3.788

Table 4 represents the variation of nonlinear frequency vs dimensionless amplitudes for simply supported square panels with movable edges for various geometries with different thickness variation parameters, using $\nu=0.3$, $\lambda=\nu^2$; Berger's approach gives unsound results for movable edges.

Numerical Results: Infinitely Long Panel (i.e., $b\to\infty$)

In this case, Eq. (24) reduces to

$$\mu_1=\frac{3m^2\pi^2}{4}+\frac{\pi^4}{16}\left\{1+m^2\left(1-\frac{6}{\pi^2}\right)\right\}+\left(\frac{a^2}{Rh_0}\right)^2\left[\frac{1536m}{\log[(1+m)/(1-m)]}+12\lambda\right]$$

$$\mu_2=-\left(\frac{a^2}{Rh_0}\right)\left[\frac{36m}{\log[(1+m)/(1-m)]}+16\lambda\right]$$

$$\mu_3=\frac{3\pi^4}{16}\frac{m}{\log[(1+m)/(1-m)]}+\frac{3\lambda\pi^4}{128}\left(9+\frac{24m}{\pi^2}\right) \quad (29)$$

Table 5 presents a comparative study of the variation of nonlinear frequency vs nondimensional amplitudes for a simply supported infinitely long panel with immovable edges for various geometries with different thickness variations using $\nu=0.3$, $\lambda=\nu^2$. In this table, results of the present study are compared with those obtained by Berger's approach.

In the case of movable edges, Eq. (26) reduces to

$$\mu_1=\frac{3m^2\pi^2}{4}+\frac{\pi^4}{16}\left\{1+m^2\left(1-\frac{6}{\pi^2}\right)\right\}+12\lambda\left(\frac{a^2}{Rh_0}\right)^2$$

$$\mu_2=-16\lambda\left(\frac{a^2}{Rh_0}\right)$$

$$\mu_3=\frac{3\lambda\pi^4}{128}\left(9+\frac{24m}{\pi^2}\right) \quad (30)$$

Table 6 presents the variation of nonlinear frequency vs dimensionless amplitudes for a simply supported, infinitely long panel with movable edges for various geometries with different thickness variations using $\nu=0.3$, $\lambda=\nu^2$; Berger's approach gives unsound results for movable edges.

Conclusions and Observations

From the present study of spherical and cylindrical shells of nonuniform thickness, the following observations may be made:

Berger's equations, although decoupled, are meaningful only for immovable edge conditions and lead to meaningless results for movable edge conditions. But the differential equations proposed in the present study are decoupled and thus simplified. It yields excellent results for movable as well as for immovable edges. It is to be noted that from a single equation, the results for both movable and immovable edge conditions can be obtained easily and accurately. This is certainly an additional advantage.

Marguerre's shallow shell equations, as used by Ramachandran for constant thickness, are in coupled form and thus require much more computational labor to solve. But these decoupled forms of differential equations of the present study are easy to handle, and accurate results can easily be obtained.

In the case of spherical shells, nonlinear frequency increases as thickness increases, whereas in the case of cylindrical shells, frequency decreases as thickness increases, due to the nature of the Gaussian curvature in the corresponding cubic equation of the cylindrical shell. From the engineering point of view, all these results are to be expected.

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